Exercise 2.2.11

(Analytical solution for charging capacitor) Obtain the analytical solution of the initial value problem

$$\dot{Q} = \frac{V_0}{R} - \frac{Q}{RC}$$

with $Q(0) = 0$, which arose in Example 2.2.2.

Solution

Here the aim is to solve the following initial value problem.

$$\frac{dQ}{dt} = \frac{V_0}{R} - \frac{Q}{RC}, \quad Q(0) = 0$$

Start by rewriting the right side.

$$\frac{dQ}{dt} = -\frac{Q - CV_0}{RC}$$

Solve this ODE by separating variables and integrating both sides.

$$\frac{dQ}{Q - CV_0} = -\frac{dt}{RC}$$

$$\int \frac{dQ}{Q - CV_0} = \int -\frac{dt}{RC}$$

$$\ln|Q - CV_0| = -\frac{t}{RC} + D \tag{1}$$

Apply the initial condition now to determine D.

$$\ln|0 - CV_0| = D = \ln|CV_0|$$

As a result, equation (1) becomes

$$\ln|Q - CV_0| = -\frac{t}{RC} + \ln|CV_0|.$$

Bring the logarithms to the left side and combine them.

$$\ln\left|\frac{Q - CV_0}{CV_0}\right| = -\frac{t}{RC}$$

Exponentiate both sides.

$$\left|\frac{Q - CV_0}{CV_0}\right| = e^{-t/RC}$$

Remove the absolute value sign by placing $\pm$ on the right side.

$$\frac{Q - CV_0}{CV_0} = \pm e^{-t/RC}$$

Choose the minus sign so that when $Q = 0$ and $t = 0$, the equation is a true statement.

$$\frac{Q - CV_0}{CV_0} = -e^{-t/RC}$$

Therefore, solving for Q,

$$Q(t) = CV_0(1 - e^{-t/RC}).$$